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Different Monotonicity Definitions in Stochastic Modelling

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Résumé

In this paper we discuss different monotonicity definitions applied in stochastic modelling and discuss the relationships between them. Obviously, the monotonicity concept depends on the relation order that we consider on the state space. In the case of total ordering the stochastic monotonicity used in stochastic modelling and the realizable monotonicity used in perfect simulation are equivalent to each other while in the case of partial order there is only an implication between them.

1. Introduction

We consider in this paper monotonicity definitions applied in different context of stochastic modelling. First of them is the stochastic monotonicity concept associated to a stochastic ordering relation. In general the considered order relation on the state space is a total ordering. However the partial order is more suitable for multidimensional models. We explain first the stochastic monotonicity for a state space endowed with at least a pre-order and study the relationships with other monotonicity definitions.

The other monotonicity definitions are related to the perfect simulation. It has been shown that if the underlying model is realizable monotone, it is possible to generate only two trajectories (sandwiching property) [2]. This concept is called realizable monotonicity. The other definition is used in a software to provide perfect simulation of queueing networks (http ://www-id.imag.fr/Logiciel/psi/). This is called event monotonicity and has been defined in more general terms in the work of Glasserman and Yao [6].

In this paper we present these definitions by emphasizing if the state space is totally ordered or not. We then compare them to give insights on the implications between them. We have considered relations between monotonicity definitions in a totally ordered state space [5]. In this case, the stochastic monotonicity and the realizable monotonicity are equivalent to each other. Therefore it is possible to construct bounding and stochastic monotone models in order to do monotone perfect simulations of systems which are not realizable monotone.

2. Different Definitions of Monotonicity

Here we present different monotonicity definitions used in stochastic modelling. First of them is the stochastic monotonicity which is defined in [1,7].

In the following example, we discuss the st-monotonicity by considering respectively a total order and then a partial order relation on the state space to show that there is no implication.

Example

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/6 & 1/3 & 0 \\ 1/2 & 1/6 & 0 & 1/3 \\ 1/2 & 0 & 1/6 & 1/3 \\ 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

First we consider a total order : $S = \{a, b, c, d\}$ and $a \preceq b \preceq c \preceq d$. We can see easily that the rows are increasing , so the matrix is stochastic monotone in the total ordering. Now we consider a partial order : $a \preceq b \preceq d$; and $a \preceq c \preceq d$. The increasing sets are $\Gamma_1 = \{a\}$, $\Gamma_2 = \{c,\ d\}$, $\Gamma_3 = \{b,\ d\}$, $\Gamma_4 = \{b,\ c,\ d\}$, $\Gamma_5 = \{a,\ b,\ c,\ d\}$. **P** is not monotone with respect to this order. For instance, for $\Gamma_3 = \{b,\ d\}$, the probability measure for row b is $1/6 + 1/3$, while this measure is $1/3$ for row d. Since $b \preceq d$, this violates the monotonicity.

Therefore we can see that the monotonicity with a total order does not imply the monotonicity with a partial order.

2.1. Monotonicity in Perfect simulation

We now present monotonicity definitions applied in perfect simulation. Let us remind that we consider that state space S is endowed with a relation order $\preceq$ which is at least a pre-order. First we will give the definition of realizable monotonicity, used in Fill's works [3].

2.1.1. Realizable monotonicity

For simulation we use the following representation of a DTMC :

* Ce travail a été réalisé avec Nihal Pekergin et Jean-Marc Vincent.

Definition 1 (Transition function)**.** *A matrix transition* $\mathbf{P}$ *can be described by a transition function* $\Phi : S \times [0,1] \to S$*, which is defined by*

$$X_{n+1} = \Phi(X_n, U_{n+1});$$

where X_n is n^{th} *observed state of the system, and* $U_{n\{n \in \mathbf{Z}\}}$ *the sequence of inputs of the system, typically a sequence of calls to a RANDOM function uniformally distributed on [0, 1].*

Definition 2 (realizable monotonicity)**.** *Let* $\mathbf{P}$ *be a transition matrix defined on state space* S*.* $\mathbf{P}$ *is said to be realizable monotone, if there exists a transition function* Φ*, defined as in definition (1), such that* Φ *preserves the order relation i.e. for all* $u \in \mathbf{U}$*, we have* $\Phi(x,u) \preceq \Phi(y,u)$*, whenever* $x \preceq y$*.*

2.1.2. Event monotonicity

Let us now present the monotonicity definition used to perform perfect simulation of finite queuing networks by software Psi2 [4].

Definition 3 (event)**.** *An event* e *is an application defined on* S*, that associates to each state* $x \in S$ *a new state denoted by* $\Phi(x,e)$*.* Φ *is called the* ***transition function*** *of the system.*

Definition 4 (Transition function)**.** *A DTMC is described by a transition function* Φ *with events :*
$\Phi : S \times \mathcal{E} \to S$*, such that for each event* $e \in \mathcal{E}$ *we have,*

$$X_{n+1} = \Phi(X_n, e)$$

where X_n is n^{th} *observed state of the system, and* X_{n+1} *is the state resulting from* X_n *upon the occurrence of the event* e_{n+1}*.*

Definition 5 (event monotonicity)**.** *The underlying model is said to be event monotone, if the transition function (def. 4) preserves the order ie. for each* $e \in \mathcal{E}$

$$\forall (x,y) \in S \;\; x \preceq y \to \Phi(x,e) \preceq \Phi(y,e)$$

There is no equivalence between the event-monotonicity (resp. the realizable monotonicity) in the total ordering and the partial ordering.

3. Stochastic monotonicity and realizable monotonicity

When the state space is totally ordered the stochastic monotonicity and the realizable monotonicity are equivalent. However the stochastic monotonicity is necessary but not sufficient for realizable monotonicity for partially ordered state spaces. [2]

Theorem 1. *When the state space is totally ordered, there is an equivalence between the stochastic monotonicity and the realizable monotonicity.*

If we consider a partial order on the state space, we have only an implication between these two monotonicity definitions.

Theorem 2. *In the case of partially ordered state spaces, if the system is realizable monotone, it is also stochastically monotone.*

The reciprocal of this implication is not true.
We can not always find a monotone transition function for a stochastic monotone system..

4. Event monotonicity and realizable monotonicity

Now we will study the relation between the event monotonicity used for perfect monotone simulation [4] , and realizable monotonicity defined by Fill [2]. We assume that the state space S is endowed by a partial order and the results obtained in this section are valid also for total order case.

Theorem 3. *If the system is event monotone, its transition matrix is realizable monotone.*

The reciprocal of this implication is not true, when the set of events is predefined. But if we add a new events in the system, we can obtain a monotone event representation for the model.

Proposition 1. *If the system is realizable monotone, it is possible to define an event monotone transition function.*

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