Mean Field Interaction Models for Computer and Communication Systems and the Decoupling Assumption

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Abstract

We consider models of N interacting objects, where the interaction is via a common resource and the distribution of states of all objects. We introduce the key scaling concept of intensity; informally, the expected number of transitions per object per time slot is of the order of the intensity. We consider the case of vanishing intensity, i.e. the expected number of object transitions per time slot is o(N). We show that, under mild assumptions and for large N, the occupancy measure converges, in mean square (and thus in probability) over any finite horizon, to a deterministic dynamical system. The mild assumption is essentially that the coefficient of variation of the number of object transitions per time slot remains bounded with N. No independence assumption is needed anywhere. The convergence results allow us to derive properties valid in the stationary regime. We discuss when one can assure that a stationary point of the ODE is the large N limit of the stationary probability distribution of the state of one object for the system with N objects. We use this to develop a critique of the fixed point method sometimes used in conjunction with the decoupling assumption.

Full text to appear in Performance Evaluation; also available on infoscience.epfl.ch

http://infoscience.epfl.ch/getfile.py?docid=16148&name=pe-mf-tr&format=pdf&version=1

Contents

Mean Field Interaction Model

- Time is discrete
- *N* objects
- Object *n* has state $X_n(t)$ 2 $\{1, \ldots, I\}$
- $(X_1(t), \ldots, X_N(t))$ is Markov
- Objects can be observed only through their state
- *N* is large, *I* is small
- Can be extended to a common resource, see full text for details

- Example 1: *N* wireless nodes, state = retransmission stage *k*
- Example 2: *N* wireless nodes, state = *k*,*c* (*c*= node class)

- Example 3: *N* wireless nodes, state = *k*,*c*,*x* (*x*= node location)

What can we do with a Mean Field Interaction Model ?

- Large *N* asymptotics
  - ¼ fluid limit
  - Markov chain replaced by a deterministic dynamical system
  - ODE or deterministic map
- Issues
  - When valid
  - Don’t want do a PhD to show mean field limit
  - How to formulate the ODE
- Large *t* asymptotic
  - ¼ stationary behaviour
  - Useful performance metric
- Issues
  - Is stationary regime of ODE an approximation of stationary regime of original system ?
  - Does this justify the “Decoupling Assumption” ?

Contents

Intensity of a Mean Field Interaction Model

- Informally:
  Probability that an arbitrary object changes state in one time slot is O(intensity)

source	[L, McDonald, Mundinger]	[Benaïm,Weibull]	[Sharma, Ganesh, Key Bordenave, McDonald, Proutière]
domain	Reputation System	Game Theory	Wireless MAC
an object is…	a rater	a player	a communication node
objects that attempt to do a transition in one time slot	all	1, selected at random among N	every object decides to attempt a transition with proba 1/N, independent of others binomial(1/N,N)¼ Poisson(1)
intensity	1	1/N	1/N

Formal Definition of Intensity

- Definition: drift = expected change to $M^N(t)$ in one time slot

$$\vec{f}^N(\vec{m}) = E\left(M^N(t+1) - M^N(t) \,\middle|\, \vec{M}^N(t) = \vec{m}\right)$$
$$= \sum_{i \neq i'} m_i P^N_{i,i'}(\vec{m})\ (\vec{e}_{i'} - \vec{e}_i)$$

- Intensity : The function $\varepsilon(N)$ is an intensity iff the drift is of order $\varepsilon(N)$, i.e.

$$\lim_{N \to \infty} \frac{\vec{f}^N(\vec{m})}{\epsilon(N)} = \vec{f}(\vec{m})$$

Vanishing Intensity and Scaling Limit

- Definition: Occupancy Measure $M^N_i(t)$ = fraction of objects in state *i* at time *t*
- There is a law of large numbers for $M^N_i(t)$ when N is large

- If intensity vanishes, i.e. limit $_{N \to \infty} \varepsilon(N) = 0$ then large *N* limit is in continuous time (ODE)
  - Focus of this presentation

- If intensity remains constant with *N*, large *N* limit is in discrete time
  [L, McDonald, Mundinger]

Contents

Convergence to Mean Field

Hypotheses

- (1): Intensity vanishes:
- (2): coefficient of variation of number of transitions per time slot remains bounded
- (3): dependence on parameters is C^1 (= with continuous derivatives)

- Theorem: stochastic system $M^N(t)$ can be approximated by fluid limit μ (t)

$$\frac{d\vec{\mu}}{d\tau} = \vec{f}(\vec{\mu})$$

drift of $M^N(t)$

source	[L, McDonald, Mundinger]	[Benaïm,Weibull]	[Sharma, Ganesh, Key Bordenave, McDonald, Proutière]
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an object is…	a rater	a player	a communication node
objects that attempt to do a transition in one time slot	all	1, selected at random among N	every object decides to attempt a transition with proba 1/N, independent of others binomial(1/N,N)¼ Poisson(1)
intensity (H1)	1	1/N	1/N
coef of variation (H2)		0	· 2
C^1 (H3)			

Exact Large *N* Statement

- Definition: Occupancy Measure
  $M^N_i(t)$ = fraction of objects in state *i* at time *t*

- Definition: Re-Scaled Occupancy measure

$$\bar{M}^N(t\,\epsilon(N)) = M^N(t)$$

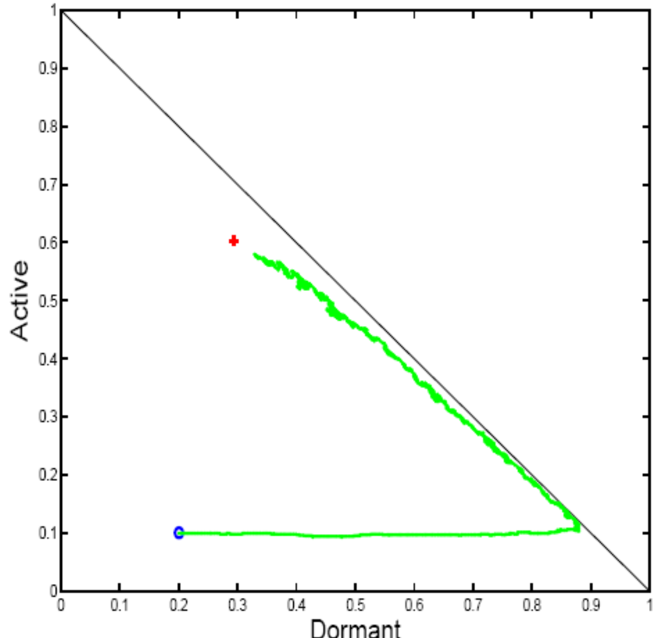
Corollary 1 *If* $M^N(0) \rightarrow \vec{m}$ *in probability [resp. in mean square] as* $N \rightarrow \infty$ *then* $\sup_{0 \le \tau \le T} \left\| \bar{M}^N(\tau) - \vec{\mu}(\tau) \right\| \rightarrow 0$ *in probability [resp. in mean square], where* $\vec{\mu}(\tau)$ *satisfies the ODE (11) and* $\vec{\mu}(0) = \vec{m}$.

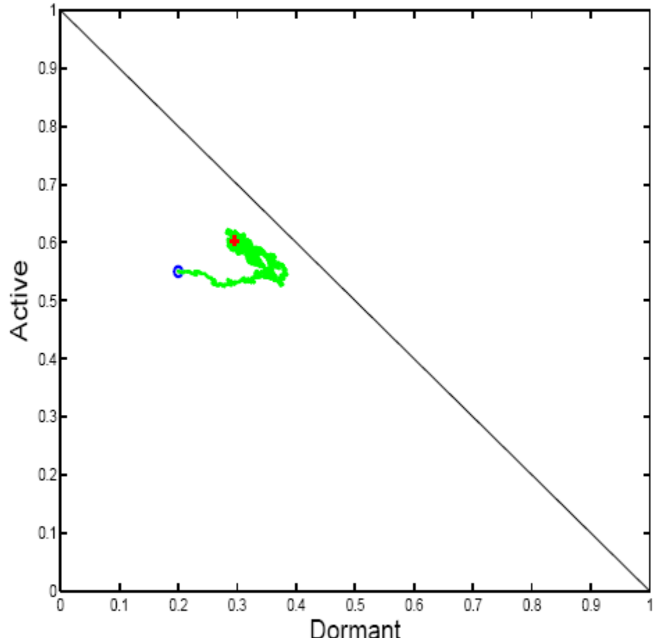
Contents

Example: 2-step malware propagation

- Mobile nodes are either
  - Susceptible
  - “Dormant”
  - Active
- Mutual upgrade
  - D + D -> A + A
- Infection by active
  - D + A -> A + A
- Recruitment by Dormant
  - S + D -> D + D
- Direct infection
  - S -> A
- Nodes may recover

- A possible simulation
  - Every time slot, pick one or two nodes engaged in meetings or recovery
  - Fits in model: intensity 1/N





Computing the Mean Field Limit

case	proba	effect on (D, A, S)
1	$D\delta_D$	$\frac{1}{N}(-1, 0, 1)$
2	$D\lambda\frac{ND-1}{N}$	$\frac{1}{N}(-2, +2, 0)$
3	$A\beta\frac{D}{h+D}$	$\frac{1}{N}(-1, +1, 0)$
4	$A\delta_A$	$\frac{1}{N}(0, -1, +1)$
5	$S(\alpha_0 + rD)$	$\frac{1}{N}(+1, 0, -1)$
6	$S\alpha$	$\frac{1}{N}(0, +1, -1)$

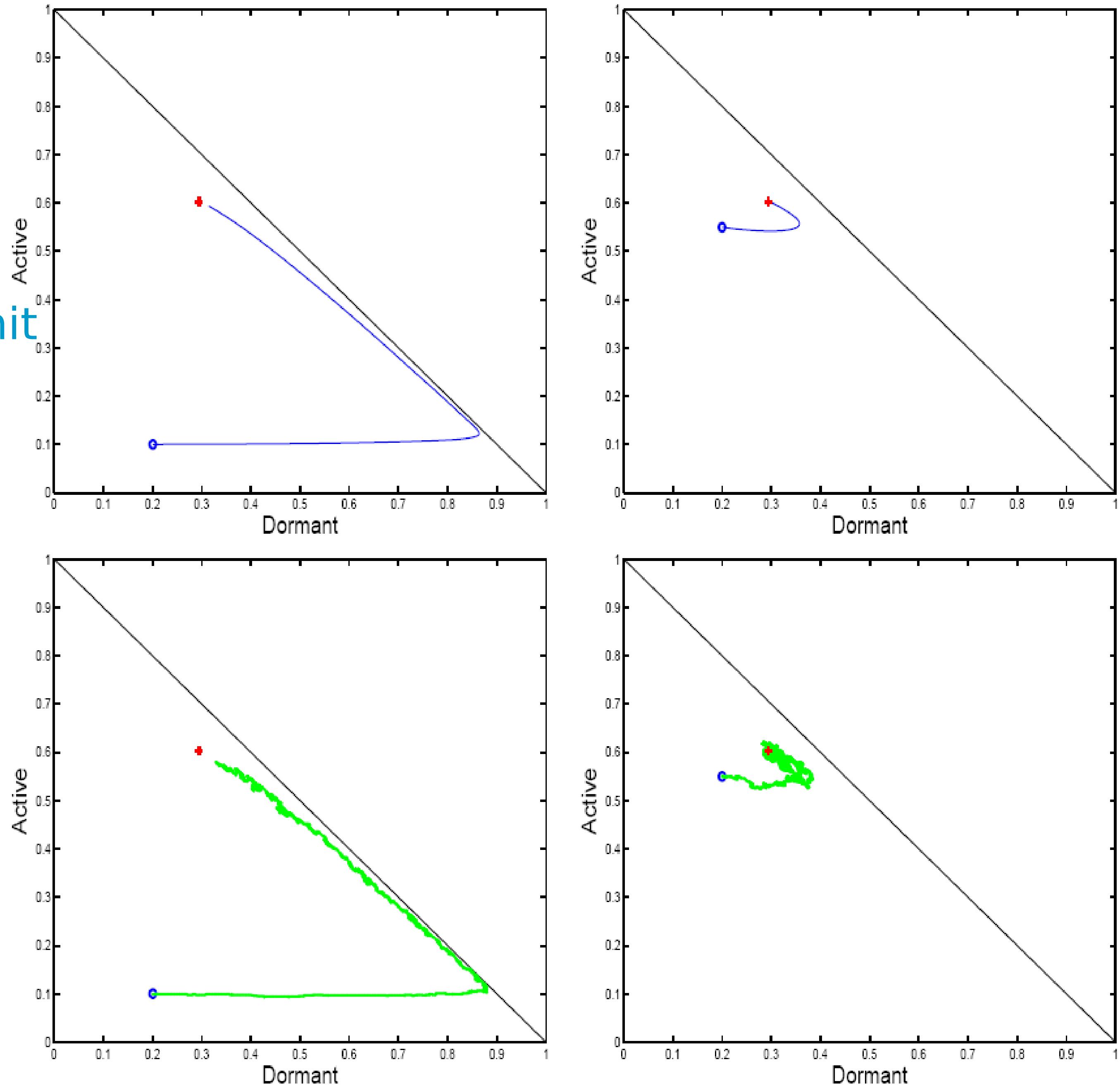
- Compute the drift of M^N and its limit over intensity

$$\vec{f}^N(D, A, S) = \frac{1}{N}\begin{pmatrix} -D\delta_D - 2D\lambda\frac{ND-1}{N} - A\beta\frac{D}{h+D} + S(\alpha_0 + rD) \\ 2D\lambda\frac{ND-1}{N} + A\beta\frac{D}{h+D} - A\delta_A + S\alpha \\ D\delta_D + A\delta_A - S(\alpha_0 + rD) - S\alpha \end{pmatrix}$$

$$\begin{pmatrix} \dot{D} \\ \dot{A} \\ \dot{S} \end{pmatrix} = \begin{pmatrix} -D\delta_D - 2D^2\lambda - A\beta\frac{D}{h+D} + S(\alpha_0 + rD) \\ 2D^2\lambda + A\beta\frac{D}{h+D} - A\delta_A + S\alpha \\ D\delta_D + A\delta_A - S(\alpha_0 + rD) - S\alpha \end{pmatrix}$$

Mean field limit
N = +1

Stochastic system
N = 1000



Contents

Propagation of Chaos

- Convergence to an ODE implies “propagation of chaos” [Sznitman, 1991]
  - This says that, for large N, any k objects are ¼ independent

Theorem 2 ([16]) *Consider a mean field interaction model with vanishing intensity and assume that the initial occupancy measures are such that the assumptions of Corollary 1 hold. Assume in addition that the collection of objects at time* 0 $(X_1^N(0), \dots, X_N^N(0)$ *is exchangeable. For any fixed* k *and* τ*:*

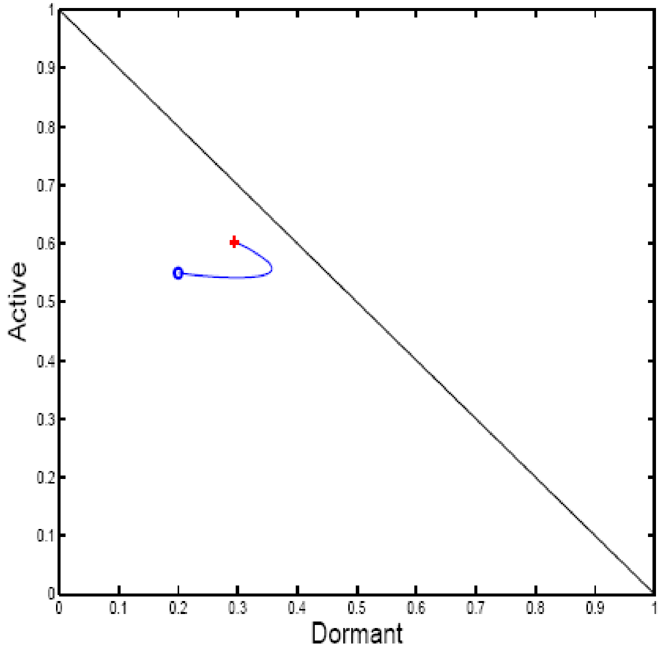
$$\lim_{N\to\infty} \mathbb{P}\left(\bar{X}_1^N(\tau) = i_1, \dots, \bar{X}_k^N(\tau) = i_k\right) = \mu_{i_1}(\tau)\dots\mu_{i_k}(\tau) \tag{18}$$

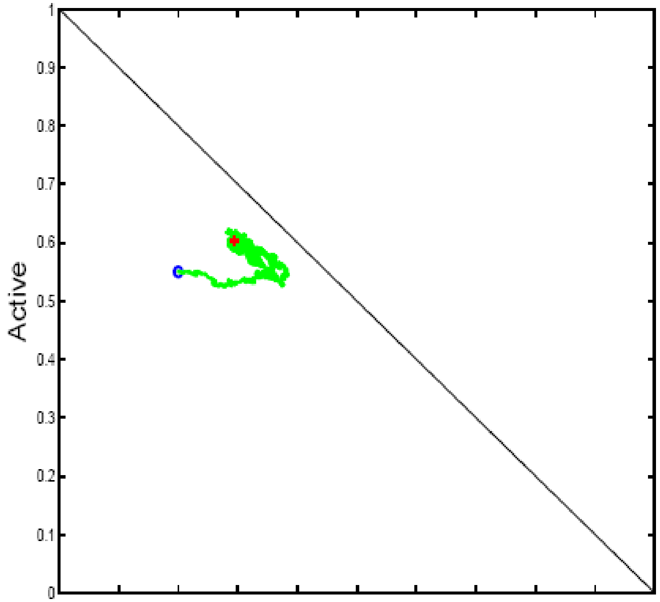
$$\mathbb{P}\left(X_1^N(t) = i_1, \dots, X_k^N(t) = i_k\right) \approx \mu_{i_1}(\frac{t}{N})\dots\mu_{i_k}(\frac{t}{N})$$

mean field limit

Example

- For large t
  - k nodes are independent
  - Prob (node n is dormant) ¼ 0.3
  - Prob (node n is active) ¼ 0.6
  - Prob (node n is susceptible) ¼ 0.1
- Propagation of chaos is also called
  - “decoupling assumption” (in computer science)
  - “mean field independence” or even simply “mean field” (in physics)





Contents

The Fixed Point Method

- Commonly used when studying communication protocols; works as follows
  - Nodes 1…N each have a state in {1,2,…,I}
  - Assume N is large and therefore nodes are independent (decoupling assumption)
  - Let m^*_i be the proba that any given node n is in state I. The vector m^* is given by the eq $\vec{F}(\vec{m}^*) = 0.$ where F is the drift.
  - Can often be put in the form of a fixed point and solved iteratively.
- Example: solve for D,S,A in

$$\begin{pmatrix} -D\delta_D - 2D^2\lambda - A\beta\frac{D}{h+D} + S(\alpha_0 + rD) \\ 2D^2\lambda + A\beta\frac{D}{h+D} - A\delta_A + S\alpha \\ D\delta_D + A\delta_A - S(\alpha_0 + rD) - S\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(with D+S+A = 1) and obtain D¼0.3, A¼ 0.6, S ¼ 0.1

Is the Fixed Point Method justified ?

- With the fixed point method we do two assumptions
  - Convergence to mean field, which is the same as decoupling assumption

$$\mathbb{P}\left(X_1^N(t) = i_1, \ldots, X_k^N(t) = i_k\right) \approx \mu_{i_1}(\frac{t}{N}) \ldots \mu_{i_k}(\frac{t}{N})$$

  - $\mu(\tau)$ converges to some m^*

When the ODE has a global attractor, the fixed point method is justified

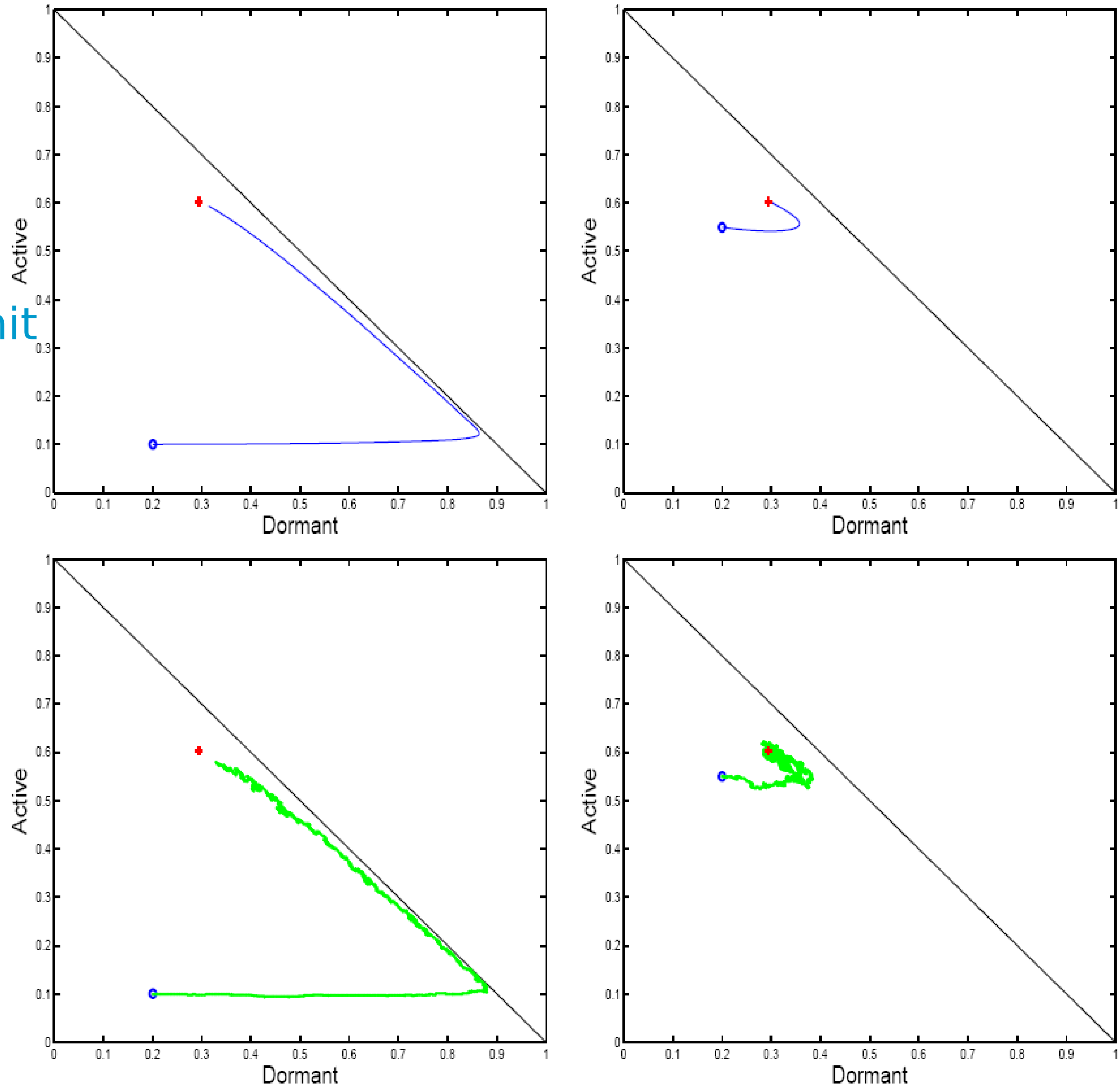
- Original system (stochastic):
  - $(X^N(t))$ is Markov, finite, discrete time
  - Assume it is irreducible, thus has a unique stationary proba η^N
- Mean Field limit (deterministic)
  - Assume (H) the ODE has a global attractor m^*
    - i.e. $\mu(\tau)$ converges to m^* for all initial conditions
- Theorem Under (H)

$$\lim_{N\to\infty} \mathbb{P}_{\eta^N}\left(X_1^N(t)=i_1, ..., X_k^N(t)=i_k\right) = m^*_{i_1}...m^*_{i_k}$$

  - i.e. the fixed point method is justified
- m^* is the unique fixed point of the ODE, defined by $F(m^*)=0$

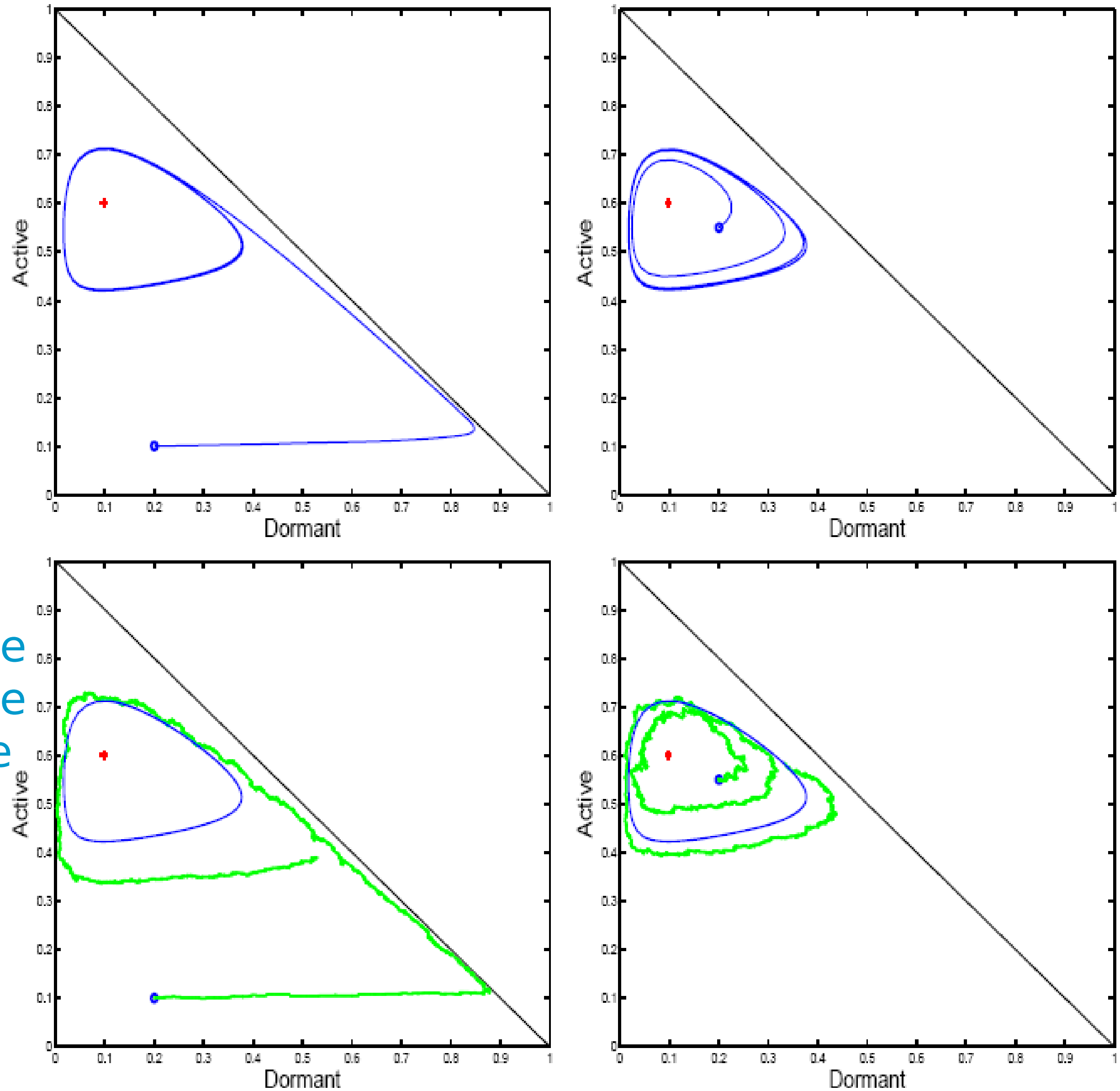
Mean field limit
N = +1

Stochastic
system
N = 1000



Assumption H may not hold, even for perfectly behaved example

Same as before
Except for one
parameter value

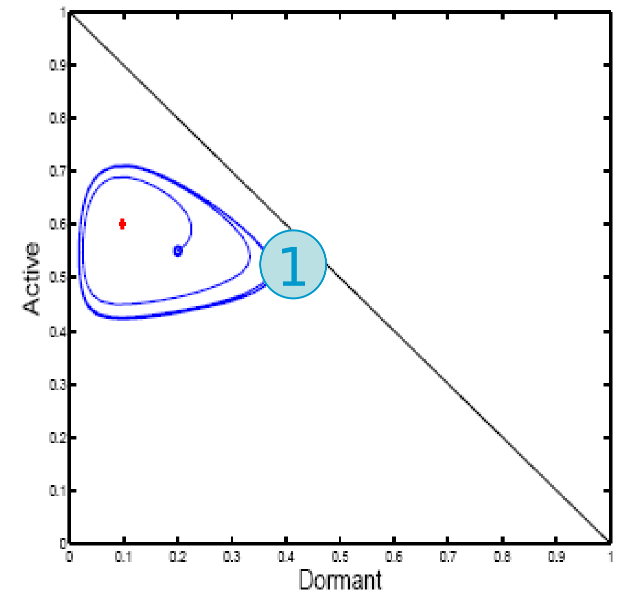


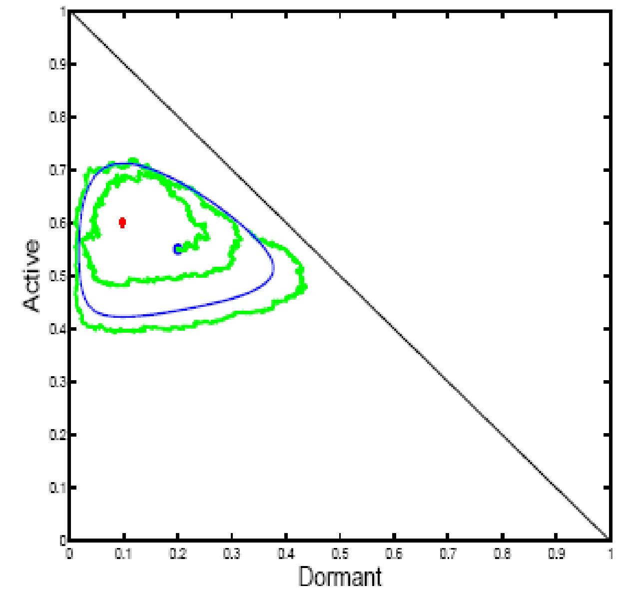
In this Example…

- $(X^N(t))$ is irreducible and thus has a unique stationary probability η^N
- There is a unique fixed point
  - $F(m^*)=0$ has a unique solution
  - but it is not a stable equilibrium
- The fixed point method would say here
  - Prob (node n is dormant) ¼ 0.1
  - Nodes are independent
- … but in reality
  - For large t, μ (t) oscillates along the limit cycle
  - Given that node 1 is dormant, it is ① most likely that μ (t) is in region

$$\mathbb{P}(X_2 = \mathrm{d} | X_1 = \mathrm{d}) > \mathbb{P}(X_2 = \mathrm{d} | X_1 = \mathrm{a})$$

  - Nodes are not independent
- Fixed Point Method does not apply





Be careful when mixing decoupling assumption and stationary regime

- Decoupling assumption holds at any time t

$$\lim_{N\to\infty} \mathbb{P}_{\eta^N}\left(X_1^N(t)=i_1,\ldots,X_k^N(t)=i_k \,\middle|\, M^N(t)=\vec{m}(t)\right)=m_{i_1}(t)\cdots m_{i_k}(t)$$

- It may not hold in the stationary regime
  - k nodes are *not* independent

- It does hold in the stationary regime if the ODE that defines the mean field limit has a global attractor

$$\vec{m}(t)\longrightarrow\vec{m}^*$$

Example: Bianchi's Formula

- Example: 802.11 single cell
  - m_i = proba one node is in backoff stage I
  - β = attempt rate
  - γ = collision proba

$$\frac{dm_0}{d\tau} = -m_0 q_0 + \beta(\vec{m})\left(1 - \gamma(\vec{m})\right) + q_K m_K \gamma(\vec{m})$$

$$\frac{dm_i}{d\tau} = -m_i q_i + m_{i-1} q_{i-1} \gamma(\vec{m}) \qquad i = 1, \ldots, K$$

$$\beta(\vec{m}) = \sum_{i=0}^{K} q_i m_i$$

$$\gamma(\vec{m}) = 1 - e^{-\beta(\vec{m})}$$

Solve for Fixed Point:

$$m_i = \frac{\gamma^i}{q_i} \frac{1}{\sum_{k=0}^{K} \frac{\gamma^k}{q_k}}$$

$$\gamma = 1 - \mathrm{e}^{-\beta}$$

$$\beta = \frac{\sum_{k=0}^{K} \gamma^k}{\sum_{k=0}^{K} \frac{\gamma^k}{q_k}}$$

Bianchi's Formula is not Demonstrated

- The fixed point solution satisfies "Bianchi's Formula" [Bianchi]

$$\gamma = 1 - e^{-\beta}$$

$$\beta = \frac{\sum_{k=0}^{K} \gamma^k}{\sum_{k=0}^{K} \frac{\gamma^k}{q_k}}$$

- Another interpretation of Bianchi's formula [Kumar, Altman, Moriandi, Goyal]

β = nb transmission attempts per packet/ nb time slots per packet

assumes collision proba γ remains constant from one attempt to next

- Is true if, in stationary regime, m (thus γ) is constant i.e. (H)

- If more complicated ODE stationary regime, not true

- (H) true for $q_0 < \ln 2$ and K= 1 [Bordenave,McDonald,Proutière] and for K=1 [Sharma, Ganesh, Key]: otherwise don't know

Contents

Generic Result for Stationary Regime

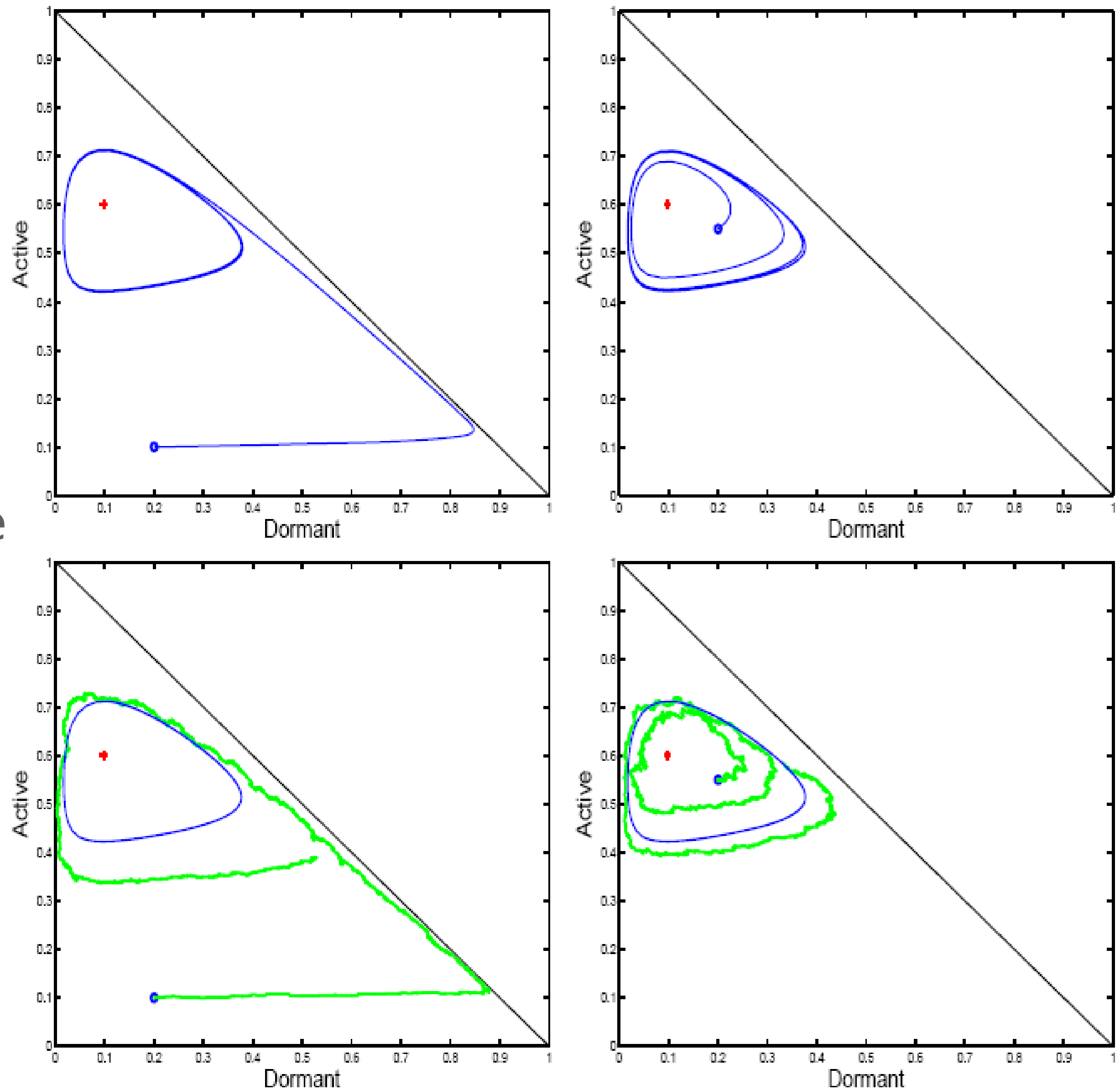
- Original system (stochastic):
  - $(X^N(t))$ is Markov, finite, discrete time
  - Assume it is irreducible, thus has a unique stationary proba ν^N
  - Let ϖ^N be the corresponding stationary distribution for $M^N(t)$, i.e. $P\left(M^N(t)=(x_1,\ldots,x_I)\right) = \varpi^N(x_1,\ldots,x_I)$ for x_i of the form k/n, k integer
- Theorem

Theorem 3 *The support of any limit point of ϖ^N is a compact set included in the Birkhoff center of Φ.*

Birkhoff Center: closure of set of points s.t. m2 ω (m)
Omega limit: ω (m) = set of limit points of orbit starting at m

- Here:
  Birkhoff center = limit cycle fixed point
- The theorem says that the stochastic system for large N is close to the Birkhoff center,

  i.e. the stationary regime of ODE is a good approximation of the stationary regime of stochastic system



Conclusion

- Convergence to Mean Field:
  - We have found a simple framework, easy to verify, as general as can be
  - No independence assumption anywhere
  - Can be extended to a common resource – see full text version

- Essentially, the behaviour of ODE for t ! +1 is a good predictor of the original stochastic system

- … but original system being ergodic does not imply ODE converges to a fixed point

Correct Use of Fixed Point Method

- Make decoupling assumption
- Write ODE
- Study stationary regime of ODE, not just fixed point
- If there is a global attractor, fixed point is a good approximation of stationary prob for one node and decoupling holds in stationary regime

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