Bounding large Markov chains using stochastic comparison and censoring techniques *

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1. Introduction

Modeling systems with huge Markov chain is still a hard problem when the chain does not exhibit some regularity or symmetry which allow analytical techniques or lumping. Stochastic comparison technique by which both transient and steady-state bounding distributions can be computed, lets to overcome this problem. In this work, we present a new approach that shows how we can compute stochastic bounds using the Censored Markov chain (CMC) and how we can deal with large Markov chains. The proposed approach may be useful to provide steady-state and transient rewards and the first passage time as well. We present here the key idea of algorithms recently developed.

2. CMCs and Stochastic Bounds

Consider a discrete time irreducible Markov chain $\{X_n : n = 1, 2, \ldots\}$ with finite state space S. Suppose that $S = E \cup E^c$, $E \cap E^c = \emptyset$. Suppose that the successive visits of X_n to E take place at time epochs $0 < n_1 < n_2 < \ldots <$. Then the chain $\{X_t^E = X_{n_t}, t = 1, 2, \ldots\}$ is called the censored chain (CMC) with censoring set E [5]. Let Q denote the transition matrix of chain X_n. Consider the partition of S to obtain a block description of Q :

$$Q = \begin{pmatrix} Q_E & Q_{EE^c} \\ Q_{E^cE} & Q_{E^c} \end{pmatrix} \begin{array}{c} E \\ E^c \end{array} \qquad (1)$$

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The CMC only watches the chain when it is in E. The matrix of the CMC is (Th. 2 in [5]) :

$$S_E = Q_E + Q_{EE^c} \left(\sum_{i=0}^{\infty} (Q_{E^c})^i \right) Q_{E^cE} \qquad (2)$$

Assume that (Q_{E^c}) does not contain any recurrent class, the fundamental matrix is $\sum_{i=0}^{\infty}(Q_{E^c})^i = (I - Q_{E^c})^{-1}$. CMCs have also been called restricted or watched Markov chains. Note that it is not necessary here that the chain is ergodic and we can study the absorbing time. In many problems Q can be large and so it is difficult to compute $(I - Q_{E^c})^{-1}$ to finally get S_E. Deriving bounds from Q_E and some information on the other blocks without computing S_E is therefore an interesting alternative approach. We have developed several algorithms to compute bounds for CMCs : (1) Truffet's algorithm [4] ; (2) many algorithms based on graphs and paths ; (3) DPY algorithm that we have already presented in [1] and which is based on blocks Q_E and Q_{E^cE}. In the following we restrict ourselves due to the space limitation to the first two types of algorithms.

2.1. Bounds

Consider two probability distributions p and q, we say that p is smaller than q in the strong stochastic sense ($p \leq_{st} q$) iff $\sum_{j=k}^{n} p_j \leq \sum_{j=k}^{n} q_j$, $1 \leq k \leq n$. It is known that monotonicity [2] and comparability of the transition probability matrices yield sufficient conditions for the stochastic comparison of Markov chains and their steady-state distributions. Vincent's algorithm [2] is the simplest solution to obtain a monotone upper bounding matrix of a stochastic matrix. To build a monotone upper bound of Q_E (which is only substochastic), Truffet's method consists in the following 2 steps : first add the slack probability in the last column of Q_E to make it stochastic and then apply Vincent's algorithm to obtain a monotone upper bound T_E to S_E. Let $T(M)$ be the stochastic matrix obtained when we apply Truffet's method on substochastic matrix M. The methods used in this paper are justified by the following theorem whose proof is given in [3] :

Theorem 1 *Let L_E be an element-wise lower bound to S_E, $L_E \leq S_E$. Then $S_E \leq_{st} T(L_E)$ and for any substochastic matrix $L_E \leq M_E \leq S_E$ we have $S_E \leq_{st} T(M_E) \leq_{st} T(L_E)$.*

Clearly Q_E is an element-wise lower bound of S_E and the theorem generalizes Truffet's method. It also states that the more accurate the element-wise lower bound of S_E, the more accurate the stochastic upper bound of S_E. To find a better lower bound than Q_E, we must consider again the definition of the transition matrix for a CMC (Eq. 2). The fundamental matrix clearly has a sample-path structure which can be used to obtain more accurate bounds.

Remark 1 *$(\sum_{i=0}^{\infty}(Q_{E^c})^i)[j,k]$ is the sum of all probability of paths entering in E^c from j and leaving it after an arbitrary number of visits inside E^c from k.*

We only need to add some paths instead of generating all of them because we need element-wise lower bounds of the fundamental matrix. This is the main idea of the approach based on paths. We have adapted several well-known graph algorithms to find some paths and compute their probability. The first passage time bound is also justified by this path structure of the fundamental matrix.

Remark 2 *We want to compute the first passage time distribution of state j in E when the initial state k is in E. In the CMC, all paths going through E^c appear with smaller lengths. Thus the passage time in the CMC provides a stochastic lower bound of the real passage time.*

2.2. Algorithms

The algorithms must find some paths which are summed up in the fundamental matrix. We have developed several algorithms and data structures to deal with paths exploration. The aim is to deal with chains which are so large that the transition matrix does not fit in memory. The first step is to obtain the states and transitions of the chain from some specifications. Transitions are described by evolution equations of states with events. We proceed by a Breadth-First Search from a chosen initial state to generate the set E of states. Each transition is associated to an event which is described by a probability that may be state-dependent and by the transitions it triggers for each of the states. We always assume that matrix Q_E fits in memory with sparse format. But we also have several algorithms and data-structures to deal with set E^c and the three other blocks when they fit or not in memory. The theory and the algorithms are presented in [3].

The first step is to remove the single loops because they do not help to find the first path from E to E going through E^c. The loops will be added at the end of the algorithm to generate a set of paths and increase the lower bound of the probability. The two basic techniques are Breadth-First Search and Shortest Path algorithms. Breadth-First Search algorithm generates all the paths of length smaller than d while Shortest Path algorithm gives the path with the higher probability when the cost of link (a, b) is $|\log(Q(a, b))|$. We add a constraint on the length to avoid paths with a very large number of states (i.e. d). Remember that we compute the shortest path according to a cost function but not the path with the shortest hop number. We compute the probability of the paths we have found and add it to the corresponding elements of Q_E to obtain an element-wise lower bound L_E to S_E.

2.3. An example

We present a rather abstract model to give some results and time measurements. We consider a set of N resources : they can be operational or faulty. We have two types of faults (hard/soft). The fault arrivals follow independent Poisson processes with rate $\lambda_h = 0.0001$ and $\lambda_s = 0.5$. The distribution of times to fix a fault are exponential with rate $\mu_h = 0.02$ and $\mu_s = 1$ except when all the resources are faulty. In that case, the repairman can speed up the fixing and with rate $\mu = 1$ all the resources are repaired. Note that the considered chain is not NCD because of the transitions with rate μ. We present in Table 1, the probability p to have N resources operational, the upper bound on this probability and the time T (in second) to compute them. Numerical examples are obtained by gathering in set E the states with 0 hard error. We can see that computation times are drastically reduced using bounding approach. It also provides results when exact analysis fails ($N = 10000$).

model size		Exact		Bound	
N	space size	T	p	T	p
100	5151	1.57	3.61e-6	0.17	4.12e-6
500	125751	168.47	7.51e-7	0.91	8.71e-7
1000	501501	603.14	4e-7	2.31	4.80e-7
10000	50015001	-	-	123.01	2.89e-7

Bibliographie

1. T. Dayar, N. Pekergin, and S. Younès. *Conditional Steady-State Bounds for a Subset of States in Markov Chains*, *SMCTools*, ACM Press, 2006.
2. J.M. Fourneau and N. Pekergin. *An algorithmic approach to stochastic bounds*, LNCS 2459, Performance evaluation of complex systems : Techniques and Tools, pp. 64-88, 2002.
3. J.M. Fourneau, N. Pekergin, and S. Younès. *Censoring Markov Chains and Stochastic Bounds*, Epew, pp 213-227, 2007.
4. L. Truffet. *Near Complete Decomposability : Bounding the error by a Stochastic Comparison Method*, Ad. in App. Prob., Vol. 29, pp.830-855, 1997.
5. Y. Q. Zhao, D. Liu. *The Censored Markov chain and the Best Augmentation*, J. App. Prob., Vol.33 pp. 623-629, 1996.